8. Modalanalyse, Eigenwertprobleme

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Notiztitel 05.09.2014

gewöhnliche Eigenwertprobleme

Vor.: $k = 0;\; K = 0$ $\qquad f = 1 \qquad\qquad f > 1$

Dgl. $\qquad m\ddot q + cq = 0 \;\;|\div m \qquad M\ddot{\bar q} + C\bar q = \bar 0 \;\;|\cdot M^{-1}$

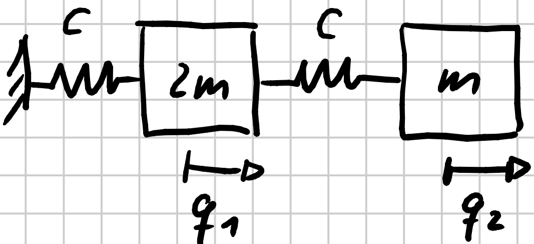
Normalform: $\qquad \ddot q + \frac{c}{m} q = 0 \qquad M^{-1}M\ddot{\bar q} + M^{-1}C\bar q = \bar 0$

1\. $\ddot q + \omega_0^2 q = 0 \qquad \underline{E}\ddot{\bar q} + C^*\bar q = \bar 0$ ✓

Eigenkreisfreq: $\omega_0 = \sqrt{\frac{c}{m}}$ $\qquad$ Frage: $C^* \overset{?}{=} \omega_0^2$?

Matrix, Skalar -> Vektor

Bsp. $f = 2$



Welche Eigenfrequenzen besitzt dieses System?

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot q_1 \\ \ddot q_2 \end{bmatrix} + \begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \bar 0 \;;\quad M\ddot{\bar q} + C\bar q = \bar 0$$

Ziel: $E\ddot{\bar q} + C^*\bar q = \bar 0$

(1) $M^{-1}M = E$

$$M = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} ;\quad M^{-1} = \frac{1}{\det M} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} = \frac{1}{2m^2} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} = \begin{bmatrix} \frac{1}{2m} & 0 \\ 0 & \frac{1}{m} \end{bmatrix}$$

$$C^* = M^{-1}C$$

$$C^* = \begin{bmatrix} \frac{1}{2m} & 0 \\ 0 & \frac{1}{m} \end{bmatrix} \cdot \begin{bmatrix} 2c & -c \\ -c & c \end{bmatrix} = \begin{bmatrix} c/m & -c/2m \\ -c/m & c/m \end{bmatrix} \overset{?}{=} \omega_0^2$$

?

korrekte Lösung für $f = n$

$$E\ddot{\bar q} + C^*\bar q = \bar 0 \;=\; E\ddot{\bar q} + A\bar q = \bar 0 \qquad C^* = A$$

Ansatz: $\bar q = \bar\psi e^{j\omega_0 t}$ $\qquad \bar\psi = [\psi_1, \psi_2, \dots \psi_n]^T$

$$\left.\begin{aligned} \dot{\bar q} &= j\omega_0 \bar\psi e^{j\omega_0 t} \\ \ddot{\bar q} &= -\omega_0^2 \bar\psi e^{j\omega_0 t} \end{aligned}\right\} \text{ in Dgl.-System}$$

$$E\left(-\omega_0^2 \bar{\psi} e^{j\omega_0 t}\right) + A \bar{\psi} e^{j\omega_0 t} = \bar{0}$$

$$\left(A - \omega_0^2 E\right) \bar{\psi} e^{j\omega_0 t} = \bar{0} \quad ; \quad \lambda := \omega_0^2 \quad ; \quad \lambda \text{ - Eigenwert}$$

Vor: $\bar{\psi} \neq \bar{0}$; $e^{j\omega_0 t} \neq 0$ $\qquad$ $\bar{\psi}$ - Eigenvektor

$$\boxed{\left(A - \omega_0^2 E\right) = \bar{0}}$$ gewöhnliches Eigenwertproblem

$(A - \lambda E) = \bar{0}$ $\quad$ math. Sachverhalt

Bsp. $f = 2$; $A = C^*$

$$\left(C^* - \lambda E\right) = \bar{0} \quad \rightarrow \quad \det\left(C^* - \lambda E\right) = 0$$

$()$

$$\begin{bmatrix} \frac{c}{m} & -\frac{c}{2m} \\ -\frac{c}{m} & \frac{c}{m} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \left(\begin{bmatrix} \frac{c}{m} - \lambda & -\frac{c}{2m} \\ -\frac{c}{m} & \frac{c}{m} - \lambda \end{bmatrix}\right)$$

$\det()$

$$\left(\frac{c}{m} - \lambda\right)\left(\frac{c}{m} - \lambda\right) - \frac{c^2}{2m^2} = 0$$

$$\lambda^2 - 2\lambda\frac{c}{m} + \frac{c^2}{2m^2} = 0 \qquad \text{quadr. Gl.}$$

$$\lambda_{1/2} = \frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - \frac{c^2}{2m^2}} = \frac{c}{m} \pm \sqrt{\frac{c^2}{2m^2}} = \frac{c}{m} \pm \frac{c}{m}\frac{1}{\sqrt{2}}$$

$$\lambda_{1/2} = \frac{c}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right) \quad ; \quad \lambda = \omega_0^2$$

$$\underline{\omega_{01}^2 = \frac{c}{m}\left(1 + \frac{1}{\sqrt{2}}\right)} \quad ; \quad \underline{\omega_{02}^2 = \frac{c}{m}\left(1 - \frac{1}{\sqrt{2}}\right)} \quad ; \quad \begin{aligned} \omega_{01} &= +\sqrt{\omega_{01}^2} \\ \omega_{02} &= +\sqrt{\omega_{02}^2} \end{aligned}$$

für $f = 2$ gibt es genau zwei Eigenkreisfreq.
ω_{01} und ω_{02}